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VIBRATIONS OF A LINEAR OSCILLATOR WITH GENERALIZED FRICTION

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A mathematical model of a special class of vibration isolation systems is investigated. The model contains formal (generalized) functions and reduces to the successive solution of boundary-value problems for differential equations with coupling conditions, ultimately yielding transcendental equations that can be solved numerically. Analytical and numerical solutions for autonomous systems are obtained, providing a means for the solution of problems in choosing the right parameters for damping systems to satisfy specified conditions for the normal operation of such systems in the presence of transient or abrupt inertial forces.

In certain structural units of kinematic vibration isolation systems, friction is introduced, obeying the Coulomb law

$$F = -f(x)N\,\mathrm{sgn}\,\dot{x}(t), \tag{1}$$

where $x(t)$ is the unknown relative displacement, $f(x)$ is the sliding (rocking) friction coefficient, which is variable in general, and $N = mg$ is the normal reaction of an oscillator of mass m and stiffness c in the gravitational field, so that the equation of motion of the oscillator has the form

$$m\ddot{x} + mgf(x)\,\mathrm{sgn}\,\dot{x} + cx = -m\tilde{W}(t), \tag{2}$$

where $\tilde{W}(t)$ is the translational acceleration in the direction of motion. The simplest type of damper provides a rocking or sliding friction force of constant magnitude, i.e., $f(x) = const$, but a damper can be constructed with "point" rocking friction $f(x) = \tilde{\Delta}\delta(x)$ or, in the general case, generalized Coulomb friction

$$f(x) = f + \tilde{\Delta}\delta(x), \tag{3}$$

where $\tilde{\Delta}$ is expressed in length units, and $\delta(x)$ is the Dirac delta function.

The oscillator equation of motion therefore has the form

$$\ddot{x} + g\left(f + \tilde{\Delta}\delta(x)\right)\mathrm{sgn}\,\dot{x} + \omega^2 x = -\tilde{W}(\lambda t), \tag{4}$$

where $\omega^2 = c/m$ and λ is the frequency parameter of the external excitation. This equation is the main topic of the present article, subject to the initial conditions $x(0) = 0$, $\dot{x}(0) = V_0$.

Introducing dimensionless variables with a certain reference length H:

$$\bar{x} = \frac{x}{H}, \quad \tau = \sqrt{\frac{g}{H}}\,t, \quad \dot{x} = \sqrt{gH}\,\bar{x}'(\tau), \quad \omega^2 = \frac{g}{H}p^2,$$

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$$W=\frac{\tilde{W}}{g},\quad \varepsilon^2=\frac{g}{H\lambda^2},\quad \Delta=\frac{\tilde{\Delta}}{H}, \tag{5}$$

we can convert Eq. (4) to the dimensionless form

$$\bar{x}''(\tau)+\left(f+\Delta\delta(\bar{x})\right)\operatorname{sgn}\bar{x}'(\tau)+p^2\bar{x}(\tau)=-W(\tau/\varepsilon) \tag{6}$$

with the initial conditions

$$\bar{x}(0)=0,\quad \bar{x}'(0)=\frac{V_0}{\sqrt{gH}}=\bar{V}_0\,. \tag{7}$$

Specified in the form of (6), the equation of motion does not have a numerical solution and must therefore be transformed into a problem amenable to subsequent solution by known methods. To that end, we multiply Eq. (6) by $\bar{x}'(\tau)$ and integrate over a short time interval $[\tau_k^+, \tau_k^-]$, where $\bar{x}(\tau_k) = 0$. Integration yields

$$\frac{1}{2}\bar{x}_+'^2-\frac{1}{2}\bar{x}_-'^2+\Delta=0, \tag{8}$$

i.e., at the instant of passage through the point $\bar{x} = 0$ the magnitude of the oscillator velocity suffers a discontinuity and is subsequently equal to

$$\left|\bar{x}_+'\right|=\sqrt{\bar{x}_-'^{\,2}-2\Delta}\,.$$

The sign of the velocity must be chosen to satisfy the condition

$$\operatorname{sgn}\bar{x}_+'\left(\tau_k\right)=\operatorname{sgn}\bar{x}_-'\left(\tau_k\right),$$

otherwise the motion will come to a complete halt.

The following definitions have been used in the derivation of Eq. (8) [1]:

$$\int_a^b\frac{dg(x)}{dx}dx=g(b)-g(a);$$

$$\int_a^b g(x)\delta\left(f(x)\right)dx=\sum_{k=1}^{N}g\left(x_k\right)\frac{1}{\left|f'\left(x_k\right)\right|},\quad \text{where}\quad f\left(x_k\right)=0,\quad a\le x_k\le b.$$

We therefore arrive at the solution of a series of initial-value problems on the intervals $[\tau_{k-1}, \tau_k]$ (k = 1, 2, ...), $\tau_0 = 0$:

$$\bar{x}_k''+f\operatorname{sgn}\bar{x}_k'+p^2\bar{x}_k=-W(\tau/\varepsilon);$$

$$\bar{x}_k\left(\tau_{k-1}\right)=0,\quad \bar{x}_k'\left(\tau_{k-1}\right)=\sqrt{\bar{x}_{k-1}'^2-2\Delta}\operatorname{sgn}\bar{x}_{k-1}'\left(\tau_{k-1}\right),\quad (k\ge 2);$$

$$\bar{x}_1(0)=0,\quad \bar{x}_1'(0)=\bar{V}_0,\quad (k=1). \tag{9}$$

The limits of the integrals (τ_{k-1}, τ_k) are determined from the condition $\bar{x}_k(\tau_k) = 0$.

A solution of Eqs. (9) can be obtained in analytical form on intervals $(\tau_{k-1}, \bar{\tau}_k)$, $(\bar{\tau}_k, \tau_k)$ in which the sign of the velocity does not change, where $\bar{x}'(\bar{\tau}_k) = 0$, but the values of the points τ_k, $\bar{\tau}_k$ can only be found numerically, so that the analytical form of the solution does not have any particular advantages over a direct numerical approach such as (e.g.) the Runge–Kutta method.

Free Vibrations. In this case, we have $W(\tau/\varepsilon) = 0$, so that vibrations exist only as the result of a finite initial velocity $V_0 \neq 0$, which determines the length parameter $H = V_0/\omega$.

The exact solution for $f = 0$,

$$x_k(t) = \frac{1}{\omega}\sqrt{V_0^2 - 2(k-1)\tilde{\Delta} g}\,\sin p\tau = \frac{V_0}{\omega}\sqrt{1 - 2(k-1)\frac{\tilde{\Delta} g}{V_0^2}}\,\sin \omega t,$$

indicates the number of cycles $x_k(t)$ $(k \le N)$, i.e., the time of motion T to a complete stop:

$$N = \left[\frac{V_0^2}{2g\tilde{\Delta}}\right] + 1, \quad T = \frac{\pi N}{\omega}$$

($[a]$ denotes the integral part of the number a), and the amplitude of the kth cycle ($k = 1, 2, \ldots, N$) is

$$A_k = A_0\sqrt{1 - 2(k-1)\frac{\tilde{\Delta} g}{V_0^2}}; \quad A_0 = \frac{V_0}{\omega},$$

so that the square of the amplitude decreases in the time π/ω by a constant amount $2\tilde{\Delta} g/\omega^2$.

An analytical solution for free vibrations without “point” friction ($\tilde{\Delta} = 0$) can be obtained analogously. The solution of the problem has been published in many books, for example [2], so that only the final implications of the solution will be given here:

- A dead zone (zero motion) exists, having a width $[-fH/p^2, fH/p^2]$, so that during a “half-period” equal to π/ω in real time the vibration amplitude decreases by the amount $2fg/\omega^2$.
- For a given velocity $\bar{V}_0$, the initial vibration amplitude is equal to

$$\bar{A}_0 = \bar{x}_1(\bar{\tau}_1) = \frac{\sqrt{f^2 + (\bar{V}_0 p)^2} - f}{p^2}, \quad \tan p\bar{\tau}_1 = \frac{\bar{V}_0 p}{f} = \tan(\omega \bar{t}_1),$$

so that the number of vibration cycles N is determined by the integral part of the quotient $\left[\sqrt{f^2 + V_0^2 p^2}/2f\right]$, the total vibration time is equal to $T = (\bar{t}_1 + \pi N/\omega)$, and the vibration amplitude of the kth cycle is equal to

$$A_k = \bar{A}_0 H - 2(k-1)\frac{fg}{\omega^2}; \quad (k = 1, 2, \ldots, N).$$

Consequently, if time is measured from the instant $\bar{t}_1$, the duration of the motion with constant friction f is shorter than with “point” friction $\Delta = f$ for $\bar{A}_0 < 1$, and this result is more consistent with the real values of V_0, ω, and f. However, this conclusion does nothing to diminish the positive significance of point friction, which adds to the constant friction, whose coefficient f does not exceed 0.001.

Forced Vibrations. In view of the considerable uncertainty of the acceleration $W(\lambda t)$, it is customary to investigate the reaction of the oscillator to a harmonic excitation of the type $W = A\sin(\lambda t + \varphi)$, where the amplitude of the acceleration is given in fractions of $g : A = kg$. If we consider the resonance process $\lambda = \omega$, a numerical analysis has shown that “point” friction with a magnitude Δ without the constant friction f does not support the linear growth of resonance vibrations for all values of Δ that allow the initial process of motion. Constantly activated friction f for values of $f > \pi k/4$ supports the vibrational process without growth of the forced vibration amplitude, but then the motion is interrupted by stops, whose duration depends on the coefficient f (without "point" friction). If both mechanisms are operative, the point friction changes the threshold value of f at which resonance occurs, lowering it, but the period of the forced vibrations does not change.

In reality, even if the resonance state of harmonic excitation does occur, it has a finite duration, so that it is always possible to make the parameter ω characterizing the intrinsic time interval much smaller than the parameter λ characterizing the forced component, i.e., to make $\omega = \varepsilon\lambda$, $\varepsilon \ll 1$. Since the solution of the equation of motion has two components $x(t) = x_1(\omega t) + x_2(\lambda t) = x_1(\tau) + x_2(\tau/\varepsilon)$, the second term is nonvanishing only in the interval $0 \le \tau \le O(\varepsilon)$ and is described by the equation

$$x_2''(z) + \varepsilon^2\left[f + \Delta\delta(x)\right]\operatorname{sgn} x_2'(z) + \varepsilon^2 p^2 x_2(z) = -W(z)\varepsilon^2, \quad z = \tau/\varepsilon,$$

which means that the forced vibrations can be described to within terms $O(\varepsilon^2)$ by the equation

$$x_2''(z) = -W(z)\varepsilon^2$$

subject to the initial conditions $x_2(0) = 0$, $x_2'(0) = 0$, so that $x_1(0) = 0$, $x_1'(0) = V_0\varepsilon/\omega$, i.e.,

$$x_2(z) = -\varepsilon^2 \int_0^z W(t)(z-t)\,dt.$$

The equation for $x_1(\tau)$ is the initial equation without the right-hand side, i.e., the component $x_1(\tau)$ describes free vibrations under the influence of the "initial" velocity from the function $x_2(z)$ in the limit $z \to \infty$, for example $V_0 = kg/\lambda$, if we assume that internal vibrations originate from harmonics of the frequency λ. The initial conditions for $x_1(\tau)$ are written in the dimensionless form $\bar{V}_0 = k\varepsilon/p^2$.

In the final analysis, when the condition $\omega/\lambda \ll 1$ holds, the analysis of the vibrational process entails investigating the free vibrations of a compliant oscillator; the results can be used to design a damping system with several constraints on the behavior system as the response to an impulse.

Suppose, for example, that it is required to choose the parameters f and $\tilde{\Delta}$ of a damping system in such a way that the vibration amplitude will not exceed a certain level βH, and the motion of the system will terminate upon passing through the rest position, i.e., $x(T) = 0$, in the minimum number of free-vibration cycles. This problem is solved on the basis of the results obtained above, viz.:

1) the value of βH must be greater than the width of the dead zone:

$$\beta > \frac{f}{p^2}, \quad f < \beta p^2 = f_1;$$

2) the "initial" velocity $\bar{V}_0 = k\varepsilon/p^2$ must be greater than $\sqrt{2\Delta}$:

$$\Delta < \frac{\bar{V}_0^2}{2} = \Delta_2;$$

3) the first vibration amplitude must not be greater than βH:

$$\bar{x}_1(\bar{\tau}_1) = \frac{\sqrt{f^2 + (\bar{V}_0 p)^2} - f}{p^2} \le \beta, \quad f > \frac{\bar{V}_0^2 - (\beta p)^2}{2\beta} = f_2, \quad \bar{x}_1'(\bar{\tau}_1) = 0;$$

4) the first vibration amplitude must not be greater than twice the width of the dead zone:

$$\bar{x}_1(\bar{\tau}_1) \ge \frac{2f}{p^2}, \quad f \le \frac{\bar{V}_0 p}{\sqrt{8}} = f_3;$$

5) the velocity through the equilibrium position $\bar{x}_1(\tau_1) = 0$ must be smaller than $\sqrt{2\Delta}$:

$$\left[\bar{x}_1'(\tau_1)\right]^2 < 2\Delta,$$

$$\Delta \geq \frac{1}{2}\frac{\left(2f - \sqrt{f^2 + \left(\bar{V}_0 p\right)^2}\right)^2}{p^2} = \Delta_1(f).$$

By virtue of conditions 3) and 4), the constant sliding friction coefficient f lies in the interval $f_2 \leq f \leq f_3$ (for $f_3 < f_1$), imposing an additional constraint on the maximum displacement, since $f_3 > f_2$:

$$\beta \geq \frac{1}{\sqrt{2}}\frac{\bar{V}_0}{p}. \tag{10}$$

Consequently, once the initial design data are available – the “initial” velocity $V_0 = kg/\lambda$, the frequency parameter $p^2 = cH/mg$, and the maximum admissible deviation $x = \beta H$, satisfying the constraints (10), the damping parameters can be chosen:

$$\frac{\bar{V}_0^2 - (\beta p)^2}{2\beta} \leq f \leq \frac{\bar{V}_0 p}{\sqrt{8}}, \quad \Delta_1(f) < \Delta \leq \frac{1}{2}\bar{V}_0^2$$

to permit admissible motion of a linear oscillator, which adequately describes a very broad category of vibration isolation systems, seismic isolation in particular [3]. We note that here “point” friction is an additional condition for normal motion of the system and can be utilized to stop the vibrational process in the minimum possible time in the presence of a small constant sliding friction.

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